INVESTIGATION OF PLANE SYMMETRY IN LATTICE DESIGNS

Honors Thesis

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Abstract

The purpose of this research project is to analyze the scholarly article The Plane Symmetry Groups: Their Recognition and Notation by Doris Schattschneider. In this article, Schattschneider discusses an application of abstract algebra which is useful in art as well as crystallography: frieze groups and wallpaper groups. I was interested in pursuing this topic because it combines mathematics with its applications, particularly with my own interest in chemistry. The article provides a compiled resource of terminology and rules of these groups, but not one which was easily acceptable to undergraduate students. In my research, I elaborated on the descriptions of certain types of periodic pattern, and analyzed a few designs to prove their classification based on the rules from Schattschneiders article. I found that this resource provided a good source of rules for which mathematical proofs could be based on, and proved the classification of two different periodic plane designs.

Acknowledgements.

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1 Introduction.

In abstract algebra, we are introduced to the concept of symmetry. Obviously, as logical observers of the world around us, we have always been aware of symmetry. In art classes, we would study how one thing looks like another in science classes, we learned about why a butterfly looks the same on both of its wings. For example, Merriam Webster defines symmetry in simplest terms as: "the quality of something that has two sides or halves that are the same or very close in size, shape, and position [1]." As a mathematician, we learned a different definition of the concept of symmetry. The more mathematical definition of symmetry given by Merriam Webster is "a rigid motion of a geometric figure that determines a one-to-one mapping of itself; the property of remaining invariant under certain changes (as of orientation in space)[1]."

Throughout our standard undergraduate education, we familiarize ourselves with symmetry in regular polygons. For example, the group of symmetries of a square is commonly characterized as D_4. These symmetries consist of four rotations, two mirror reflections, and two compositions (rotation-reflection) which we recognize as a diagonal flip. This is the simplest example of plane symmetry.

When we consider more specific plane figures, as well as considering complex designs and a pattern combining a collection of different figures, we run into the concept of lattice designs in the plane. In analyzing the symmetries of these designs, we can define frieze groups and wallpaper groups. In [3], Schattschneider defines frieze groups as designs which are invariant under all multiples of just one translation and wallpaper groups as "patterns which are invariant under linear combinations of two linearly independent translations [and] repeat at regular intervals in two directions (439)."

In [3], Schattschneider studies these groups, and produces a compiled resource of terminology and rules of these groups in order to analyze the repeating designs in various periodic patterns in her case, specifically art pieces by M.C. Escher. She provides a variety of lattice designs taken

from several different sources and classifies them as each type of periodic pattern. The goal of the following paper is to elaborate on the descriptions of certain types of periodic pattern, consider a few designs and prove their classification based on the rules from [3], and go on to analyze other designs from M.C. Escher which were provided from [2].

2 Wallpaper Group Properties.

We will begin by describing the properties of wallpaper groups described by Schattenschneider in [3]. Firstly, we must define the patterns which we will be examining. The periodic patterns which we are looking at are defined by repeated identical tiles in the plane. The tile is defined to be a shape which makes up a single unit of the pattern, which we refer to as a *lattice unit*. There are five different types of lattice unit, each a different geometric shape. The lattice types are as follows: parallelogram, rectangular, rhombic, square, and hexagonal. We use these lattice units to recognize the generating tile of the pattern in order to identify the symmetry type of the design.

We will include a chart providing the requirements for each of the seventeen different plane symmetry groups, which was given in [3].

RECOGNITION CHART FOR PLANE PERIODIC PATTERNS

Type	Lattice	Highest Order of Rotation	Reflections	Non-Trivial Glide Reflections	Generating Region	Helpful Distinguishing Properties
*p*1	parallelogram	1	no	no	1 unit	
*p*2	parallelogram	2	no	no	1/2 unit	
pm	rectangular	1	yes	no	1/2 unit	
pg	rectangular	1	no	yes	1/2 unit	
cm	rhombic	1	yes	yes	1/2 unit	
pmm	rectangular	2	yes	no	1/4 unit	
pmg	rectangular	2	yes	yes	1/4 unit	parallel reflection axes
pgg	rectangular	2	no	yes	1/4 unit	
cmm	rhombic	2	yes	yes	1/4 unit	perpendicular reflection axes
*p*4	square	4	no	no	1/4 unit	
*p*4*m*	square	4	yes	yes	1/8 unit	4-fold centers on reflection axes
*p*4*g*	square	4	yes	yes	1/8 unit	4-fold centers not on reflection axes
*p*3	hexagonal	3	no	no	1/3 unit	
*p*3*m*1	hexagonal	3	yes	yes	1/6 unit	all 3-fold centers on reflection axes
*p*31*m*	hexagonal	3	yes	yes	1/6 unit	not all 3-fold centers on reflection axes
*p*6	hexagonal	6	no	no	1/6 unit	
*p*6*m*	hexagonal	6	yes	yes	1/12 unit	

CHART 3. A rotation through an angle of $360°/n$ is said to have order n. A glide-reflection is non-trivial if its component translation and reflection are not symmetries of the pattern.

In order to classify a pattern as any of the preceding symmetry groups, we need to examine 4 different properties of the design. These naming conventions were given in [3], and will be expanded on here. First, we consider the singular lattice unit which is determined to be the lattice unit for the design. We have to determine whether or not this unit is a *primitive cell* or a *centered cell*. We define a *primitive cell* as a singular lattice unit which has its centers of the highest order of rotation at its vertices.[3] For example, patterns and designs notwithstanding, a plain square in the plane would be a primitive cell because its highest order of rotation is 4, which is found at its vertices. Containing a primitive cell is the most common in each of the symmetry groups. In a few cases, however, we have a centered cell as opposed to a primitive cell. We can define a *centered cell* as one where the reflectional axes are perpendicular to the sides of the cell as opposed to right down

the center [3]. The only types of lattice unit which are defined by centered cells are the ones with rhombic lattice units, which make up two of the seventeen symmetry groups. These types of cell determine the first symbol in the naming system for the periodic symmetry groups. We name a pattern with either a **p** for primitive cell or a **c** for centered cell.

The second property which we need to examine in order to determine the classification of a pattern as a specific symmetry group is its highest order of rotation. Order of rotation is a concept that is understood by anyone with experience in an abstract algebra class, and is defined as the number of times the shape is symmetrical within a full 360 degree rotation. Using a square as an example, it has a rotation of order 4 because there are 4 points where the square will be symmetrical within a full rotation. A restriction on the orders of rotation: we can only consider rotations of order 2, 3, 4, or 6. In naming the periodic symmetry groups, the second symbol in the name is n, the highest order of rotation of the pattern.

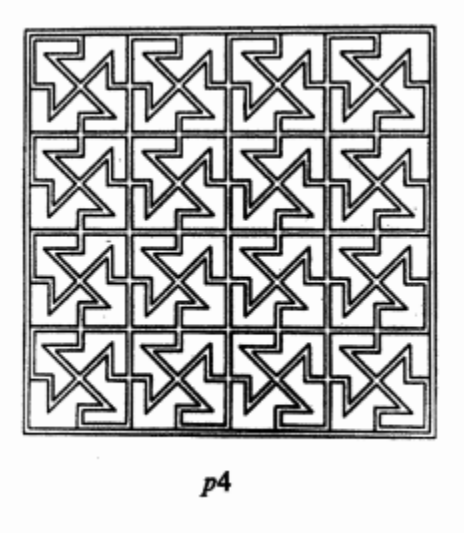
Next, we will consider the reflection symmetry axis which would be perpendicular to the x-axis; in other words, we consider the vertical reflection symmetry. If we have vertical reflection symmetry, we denote that with the symbol **m** for mirror symmetry. If we do not have mirror reflection symmetry, we move to consider a glide-reflection symmetry. In other words, is it symmetrical when we reflect and translate the pattern to another location in the plane? Then, we represent this with the symbol **g** for a glide-reflection. If there is no reflection symmetry in the pattern, but the next criteria is satisfied, we will symbolize that with a **1**.

The final criteria which we evaluate in order to name the pattern with its given symmetry group is its reflection symmetry axis at an angle α relative to the x-axis. This α is dependent on the highest order of rotation n. For $n = 1, 2$, we have that $\alpha = 180$. For $n = 4$, then $\alpha = 45$, and for $n = 3, 6$, $\alpha = 60$. We will consider the reflections at a given angle for a specific pattern in order to determine the symbol in the fourth position of the naming convention. We use the same **m, g, 1** to denote each symmetry corresponding to this angle.

So, we have a four-symbol convention for naming the symmetry group types based on the criteria described above. If there are no reflections, glide-reflections, or angle reflections, the name

will either be two or three symbols instead. We will investigate a few different lattice designs in order to visualize these criteria and prove the classification of those patterns.

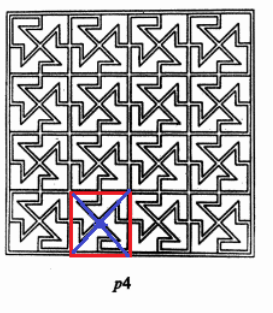
3 A Lattice Design Example.



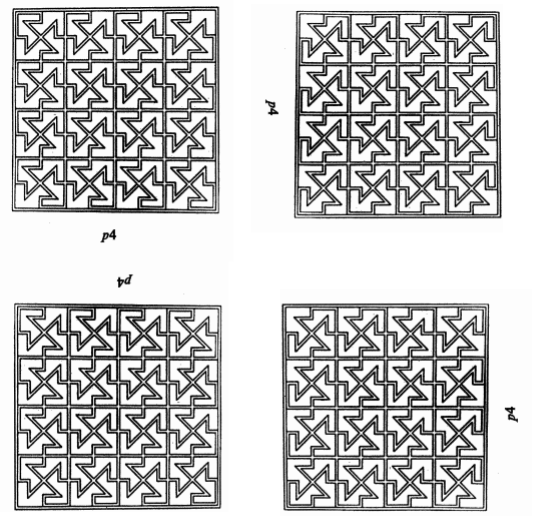
This lattice design found in [3] demonstrates an example of the symmetry group **p4**. To prove this classification, we will go through each of the group axioms noted for wallpaper symmetry groups.

Proof.

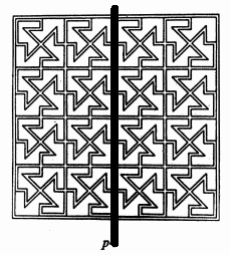
1. Notice the lattice unit that is being considered a generator for this pattern (highlighted in the next image). We want to consider whether it is a primitive cell or a centered cell. We notice that each vertex on the outside of this unit will generate the same image if we rotate the unit around that vertex. Because of this, we have that this lattice unit is a primitive cell.



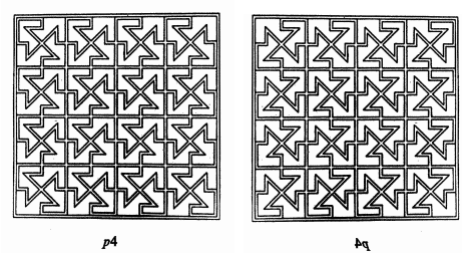
2. We want to consider the highest order of rotation for the pattern. Since the pattern is in the shape of a square, we know from abstract algebra that it will have centers of rotations at each vertex, at the midpoints of each side, and in the center of the square. Through trial and error, we have discovered: at each vertex, the order of rotation is 4; at the midpoints of each side, the order of rotation is 2; at the center of the square, the order of rotation is 4. In other words, the pattern can be rotated 0, 90, 180, 270, and 360 degrees (4 times) and still look exactly the same. Thus the highest order of rotation of this design is order 4.



3. We want to investigate whether this design has reflection symmetry. In the next image, we see an image of the design, with a thick line drawn through a vertical mirror axis.

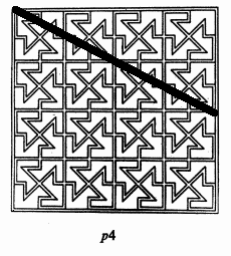


It is apparent that on either side of the line, the image is not exactly the same, thus we know that the design does not have reflection symmetry.



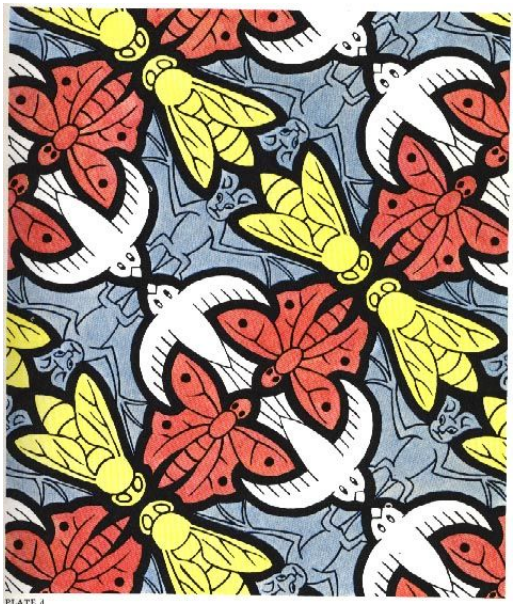
Next, we see the original design alongside the image after a translation to the right and a reflection over the vertical axis. It is apparent, again, that the image on the right and the original image are not identical; therefore this design does not have glide-reflection symmetry either.

4. We will examine whether this design has a symmetry axis along an angle α. Since this design has a highest order of rotation of order 4, we will be looking at the angle $\alpha = 60$. We are considering the left edge of the design as the x-axis for the pattern, as suggested on page 443 of the article [3]. The following image shows the design with the angle axis indicated, and we can see through observation that there is no reflection symmetry along this angle axis either.



Through this examination of the symmetry group axioms based on [3], we have shown that the image in question is classified as a **p4** symmetry group. ■

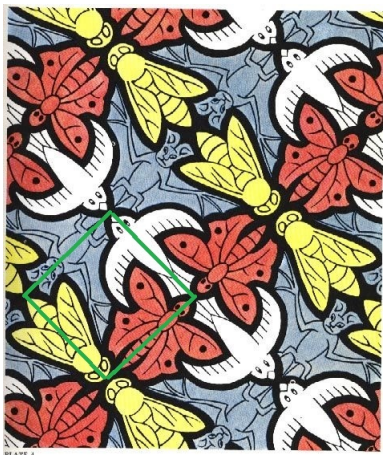
4 Another example.



This pattern is an example of one of many designs by M.C. Escher found in [2]. This pattern demonstrates an example of the symmetry group **pmm**. To prove this classification, we will again go through each of the group axioms noted for wallpaper symmetry groups.

Proof.

1. In order to determine the lattice unit which generates this pattern, we must analyze the image to find where we have repetition. Notice the lattice unit that is being considered a generator for this pattern highlighted in green outline in the next image. We want to consider whether it is a primitive cell or a centered cell. We notice that each vertex on the outside of this unit will generate the same image if we rotate the unit around that vertex. Because of this, we have that this lattice unit is a primitive cell.



2. We want to consider the highest order of rotation for the pattern. This pattern is in the shape of a rectangle. From our abstract algebra knowledge, we know that a rectangle can only have a possible order of rotation 2 - you can only rotate a rectangle twice and still receive an image symmetrical to its preimage. Once we analyze the pattern within the rectangle, we can see that there is no rotation in order to create an image symmetrical to the preimage of this pattern.

3. We want to investigate whether this design has reflection symmetry. In the next image, we see the pattern flipped over a vertical axis as though reflected in a mirror. Through observation, we are able to see that the sides of the pattern directly adjacent to one another in the middle of the original and the mirror image are exactly symmetrical, thus this pattern does have mirror symmetry.



4. Next, we need to decide whether this pattern has a symmetry axis along an angle α. This image has a highest order of rotation equal to 2, and for $n = 2$, we need to consider the symmetry along the angle $\alpha = 180$. We are considering the left edge of the design as the x-axis for the pattern, as suggested on page 443 of the article [3]. In the next image, we see an image of the design, with a green lines drawn through its various reflection axes. We can see that there are mirror reflection axes on the edges of each side of the primitive generating cell - an axis going down the middle of the bird, an axis down the middle of the fly, and axis down the middle of the moth, and an axis down the middle of the bat. So this pattern certainly has mirror reflection symmetry.



Through this examination of the symmetry group axioms based on [3], we have shown that the image in question is classified as a **pmm** symmetry group based on its primitive cell and its mirror reflective symmetries. ■

5 Conclusion.

The goal of this paper was to analyze and elaborate upon the symmetry axioms for wallpaper groups given in [3]. We have studied two different patterns, given from [3] as well as [2] in order to classify them based upon Schattschneider's naming conventions and prove that they are so based on her symmetry axioms. If allowed more time with this project, and in the future, I would like to investigate the different ways that color can play into the symmetries, and the pattern from [2] could be interesting to study for this purpose, as well as other colored illustrations from M.C. Escher. In addition, while this project focused on Symmetries in the plane, it follows that the same concepts can be studied in other spacial dimensions. For example, using some of these conventions in three dimensions, would result in analysis of crystallography which is an application of group theory in chemistry. This also is interesting to me due to my own background in chemistry, and in future analysis, I hope to be able to apply this to three dimensions as well.

References

[1] "Symmetry." *Merriam-Webster.com*. Merriam-Webster, 2016. Web. 20 November 2016.

[2] C. MacGillavry, *Fantasy and Symmetry, the Periodic Drawings of M.C. Escher*, Harry Abrams, New York, 1976.

[3] D.Schattschneider, *The Plane Symmetry Groups: Their Recognition and Notation*, American Mathematical Monthly, **85**(1978), no. 6, 439-450.